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The Entropy of a Quantum Field in a Charged Kerr Black Hole

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Abstract

We calculate the entropies of the system of classical particles and a quantum scalar field by using the brick wall method in thermal bath in a charged Kerr black hole spacetime. Their leading terms at Hartle-Hawking temperature $T_H = \kappa/(2\pi)$ are given by $S_{cl} \approx N \;\ln\left(\frac{A_b}{\epsilon^2}\right)$, and $S \approx N' \frac{A_H}{\epsilon^2}$, where A_b and A_H are the area of the box and the horizon respectively.

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1 Introduction

In 1973 Bekenstein, by comparing the black hole physics with the thermodynamics, argued that the black hole entropy is proportional to the black hole horizon area [1]. Hawking showed that the proportional coefficient is $\frac{1}{4}$ by investigating the quantum fields in a collapsing black hole space time [2]. By using the Euclidean path integral Gibbons and Hawking showed that the tree-level contribution of the gravitation action gives the black hole entropy [3]. However the exact statistical origin of the Bekenstein-Hawking entropy S_{bh} is unknown.

Recently many efforts have been concentrated to understand the statistical origin of Bekenstein-Hawking black hole entropy [4]. Frolov and Novikov argued that the black hole entropy can be obtained by identifying the dynamical degrees of freedom of a black hole with the states of all fields which are located inside the black hole [5]. Another approach is to identify the black hole entrophy S_{bh} with the entanglement entropy S_{ent} [6]. Entanglement entropy arises from ignoring the degree of freedom of a proper region of space. It is found that the entropy is proportional to the area of the boundary (horizon). However the entanglement entropy is divergent. Such divergences also arise in the brick wall method of t'Hooft [7, 8], who calculated the entropy of quantum field propagating outside the black hole. The divergences arise from the density of levels diverges due to the infinite shift of frequencies near the horizon. But It has been shown that the black hole entropy is proportional to the area after an appropriate renormalization [9]. The thermodynamic approach using the heat kernel gives the same result with the brick wall method or the entanglement entropy method [10].

In this paper we shall investigate the black hole entropy of a scalar field by the brick wall method in a charged Kerr black hole. We will also study the classical entropy of particles. Our result shows that in classical and quantum level the entropies are divergent as the system approaches to the horizon. The leading entropy of a quantum field in the Hartle-Hawking state is proportional to the area of the event horizon.

2 The Partition Function of Classical Particle

Let us consider a box containing N non-interacting particles with mass μ^2 described by a Hamiltonian $H(p,x)$ in the charged Kerr black hole space-time. We assume that the box is rotating with a constant azimuthal angular velocity Ω_0. The line element of the charged Kerr black hole spacetime in Boyer-Lindquist coordinates is given by [11, 12]

$$\begin{aligned} ds^2 &= -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ &\quad + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \qquad (1) \\ &\equiv g_{tt}\,dt^2 + 2g_{t\phi}\,dtd\phi + g_{\phi\phi}\,d\phi^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2, \qquad (2) \end{aligned}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 + e^2 - 2Mr, \tag{3}$$

and e, a, and M are charge, angular momentum per unit mass, and mass of the spacetime respectively. This spacetime has two Killing vectors: the time like Killing vector $\xi^\mu = (\partial_t)^\mu$ and the axial Killing vector $\psi^\mu = (\partial_\phi)^\mu$. We assume that $e^2 + a^2 \le M^2$. In this case the charged Kerr black hole has the event horizon at $r_{horizon} = r_+ = M + \sqrt{M^2 - a^2 - e^2}$ and has the stationary limit surface at $r_0 = M + \sqrt{M^2 - e^2 - a^2\cos^2\theta}$. Near the event horizon $r = r_+$ the metric behaves like [15]

$$ds^2 \simeq -(\kappa\rho)^2dt^2 + g_{\phi\phi}(r=r_+,\theta)\,(d\phi - \Omega_H dt)^2 + d\rho^2 + g_{\theta\theta}(r=r_+,\theta)d\theta^2, \tag{4}$$

where κ is the surface gravity of the black hole and $\rho = \int\sqrt{g_{rr}}dr$, $\Omega_H = \frac{a}{r_+^2+a^2}$. Ω_H is the angular velocity of the black hole. This form (4) is similar to the Rindler metric.

When a system with N non-interacting particles is in thermal equilibrium state at temperature $T = 1/\beta$ and is rotating with a angular velocity Ω_0 about $z-$ axis, the partition function Z_N is given by

$$Z_N = (Z)^N, \tag{5}$$

where Z is the partition function for one particle:

$$Z = \int d^3xd^3p\; e^{-\beta(\mathcal{E} - \Omega_0 p_\phi)}, \tag{6}$$

where $\mathcal{E}$, which is the energy in $\Omega_0 = 0$ frame and given by $\xi^\mu p_\mu = -p_t = \mathcal{E}$, satisfies the following

$$\mathcal{E} = \frac{1}{-g^{tt}} \left\{ -g^{t\phi} p_\phi \pm \left[(g^{t\phi} p_\phi)^2 + (-g^{tt})(\mu^2 + g^{\phi\phi} p_\phi^2 + g^{rr} p_r^2 + g^{\theta\theta} p_\theta^2) \right]^{1/2} \right\} \equiv \mathcal{E}_\pm. \quad (7)$$

Here we used $p^\mu p_\mu = -\mu^2$. $\mathcal{E} - \Omega_0 p_\phi \equiv E$ is the energy of the particle in rotating frame. We will take $\mathcal{E}_+$ because $\mathcal{E}_+$ corresponds to 4-momentum pointing toward future [13].

Note that we must restrict the system to be in the region such that ${g'}_{tt} \equiv g_{tt} + 2\Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} < 0$. In the region such that $-{g'}_{tt} > 0$ (called region I) the possible points of p_i satisfying $\mathcal{E}_+ - \Omega_0 p_\phi = E$ for a given E are located on the following surface

$$\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} + \frac{-{g'}_{tt}}{-\mathcal{D}} \left(p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{{g'}_{tt}} E \right)^2 = \left(\frac{1}{-{g'}_{tt}} E^2 - \mu^2 \right), \quad (8)$$

which is the ellipsoid, a compact surface. Here $-\mathcal{D} = g_{t\phi}^2 - g_{tt} g_{\phi\phi}$. So the density of state $g(E)$ for a given E is finite and the integrations over p_i give a finite value. But in the region such that $-{g'}_{tt} < 0$ (called region II) the possible points of p_i are located on the following surface

$$\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} - \frac{{g'}_{tt}}{-\mathcal{D}} \left(p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{{g'}_{tt}} E \right)^2 = -\left(\frac{1}{{g'}_{tt}} E^2 + \mu^2 \right), \quad (9)$$

which is the hyperboloid, a non-compact surface. So $g(E)$ diverges and the partition function Z diverges. Thus we will assume that the box is in the region I. For example, in the case of $\Omega_0 = 0$ the points satisfying ${g'}_{tt} = 0$ are on the stationary limit surface. The region of the outside (inside) of the stationary limit surface corresponds to the region I (II).

The partition function is then given by

$$\begin{aligned} Z &= \int_{region\ I} d^3x d^3p \exp\left\{ -\beta \left[(\Omega - \Omega_0) p_\phi \right.\right. \\ &\quad \left.\left. + \frac{1}{-g^{tt}} \left[(g^{t\phi} p_\phi)^2 + (-g^{tt})(\mu^2 + g^{\phi\phi} p_\phi^2 + g^{rr} p_r^2 + g^{\theta\theta} p_\theta^2) \right]^{1/2} \right] \right\}, \end{aligned} \quad (10)$$

where $\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}}$. After the integrations over p_i we get

$$\begin{aligned} Z &= 4\pi \int_{region\ I} d^3x \int_{\mu\sqrt{-{g'}_{tt}}}^{\infty} dE \frac{\sqrt{g_4}}{\sqrt{-{g'}_{tt}}} \frac{E}{-{g'}_{tt}} \left(\frac{E^2}{-{g'}_{tt}} - \mu^2 \right)^{1/2} e^{-\beta E} \\ &= \frac{4\pi}{3} \beta \int_{region\ I} d^3x \int_{\mu\sqrt{-{g'}_{tt}}}^{\infty} dE \frac{\sqrt{g_4}}{\sqrt{-{g'}_{tt}}} \left(\frac{E^2}{-{g'}_{tt}} - \mu^2 \right)^{3/2} e^{-\beta E}, \end{aligned} \quad (11)$$

where we have integrated by parts. From this expression we easily obtain the total number of state $\Gamma_{cl}(E)$ with energy less than E

$$\Gamma_{cl}(E) = \frac{4\pi}{3}\int d^3x \frac{\sqrt{g_4}}{\sqrt{-g'_{tt}}}\left(\frac{E^2}{-g'_{tt}} - \mu^2\right)^{3/2}, \tag{12}$$

which is identical to the result of ref.[16] when $\Omega_0 = a = e = 0$. This expression also can be obtained directly by investigating Eq.(8).

Let us assume that $\Omega_0 \simeq \Omega_H$, which is the angular velocity of the charged Kerr black hole. In other words the system is co-rotating with the black hole. We assume that the box is close to the horizon. Let the radial coordinates of the lower bound and the upper bound of the box be $r_+ + h$ for small h and be L respectively.

Then the leading behavior of partition function Z for $\mu = 0$ is given by

$$Z \approx \frac{8\pi}{\beta^3}\int_{r_+ + h}^{L} dr d\theta d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}}\sqrt{g_{rr}}\left(\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}\right)^{3/2} \tag{13}$$

$$\approx \frac{4\pi}{(\kappa\beta)^3}\frac{A_b}{\epsilon^2}, \tag{14}$$

where A_b is the area of the box near the horizon . ϵ is the proper distance from the horizon r_+ to $r_+ + h$:

$$\epsilon = \int_{r_+}^{r_+ + h} dr\sqrt{g_{rr}} \approx 2\left(\frac{r_+^2 + a^2\cos^2\theta}{2r_+ - 2M}\right)^{1/2}\sqrt{h} \tag{15}$$

for very small h. Therefore as the box approaches to the horizon, the partition function Z goes to be divergent quadratically in $1/\epsilon^2$. The particular point is that ϵ depends on the coordinates θ. The partition function of N particles near the horizon becomes

$$Z_N \approx \left(\frac{4\pi}{(\kappa\beta)^3}\frac{A_b}{\epsilon^2}\right)^N. \tag{16}$$

¿From this expression we obtain the leading behavior of the classical entropy

$$S_{cl} \approx N\ln\left(\frac{4\pi}{(\kappa\beta)^3}\frac{A_b}{\epsilon^2}\right), \tag{17}$$

which diverges logarithmically in ϵ as the box approaches to the horizon.

Now let us reconsider above problem in comoving coordinate which co-rotate with the box. The metric in comoving frame is

$$ds^2 = (g_{tt} + 2\Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi})dt^2 + 2(g_{t\phi} + \Omega_0 g_{\phi\phi})dtd\phi' + g_{\phi\phi}d\phi'^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2, \tag{18}$$

where we used $\phi' = \phi - \Omega_0 t$. In this frame the partition function is given by

$$Z = \int d^3x d^3p e^{-\beta E'} \tag{19}$$

where the energy E' satisfies following

$$E' = \frac{1}{-g'^{tt}} \left\{ -g'^{t\phi} p_\phi \pm \left[(g'^{t\phi} p_\phi)^2 + (-g'^{tt})(\mu^2 + g'^{\phi\phi} p_\phi^2 + g'^{rr} p_r^2 + g'^{\theta\theta} p_\theta^2) \right]^{1/2} \right\} \equiv E'_\pm. \tag{20}$$

It is easy to show that $E'_+ = \mathcal{E}_+ - \Omega_0 p_\phi$. Thus the partition function give the same value with (11). In this frame the reason that we must restrict a system to be in the region I becomes apparent. In the region II the box must move greater than the velocity of light. So it is unphysical in classical mechanics. Similar phenomena appears in the quantum field theory case.

3 A Rotating Scalar Field in the Charged Kerr Black Hole

Let us consider a minimally coupled scalar field in thermal equilibrium at temperature $1/\beta$ in the charged Kerr black hole spacetime. We assume that the scalar field is rotating with a constant azimuthal angular velocity Ω_0.

For such a equilibrium ensemble of the states of the scalar field the partition function is given by

$$Z = \sum_{n_q, m} e^{-\beta(E_q - \Omega_0 m) n_q}, \tag{21}$$

where q denotes a quantum state of the field with energy E_q and azimuthal angular momentum m. The free energy is given by

$$\beta F = \sum_m \int_0^\infty dE g(E, m) \ln \left(1 - e^{-\beta(E - m\Omega_0)} \right), \tag{22}$$

where $g(E, m)$ is the density of state for a given E and m.

To evaluate the free energy we will follow the brick wall method of 't Hooft [7]. Following the brick wall method we impose a small cut-off h such that

$$\Phi(x) = 0 \quad \text{for} \quad r = r_+ + h. \tag{23}$$

To remove the infra-red divergence we also introduce another cut-off $L \gg r_+$ such that $\Phi(x) = 0$ for $r = L$.

In the WKB approximation with $\Phi = e^{-iEt+im\phi+iS(r,\theta)}$ the Klein-Gordon equation $(\Box - \mu^2)\Phi = 0$ yields the constraint [17]

$$p_r^2 = \frac{1}{g^{rr}}\left[-g^{tt}E^2 + 2g^{t\phi}Em - g^{\phi\phi}m^2 - g^{\theta\theta}p_\theta^2 - \mu^2\right], \tag{24}$$

where $p_r = \partial_r S$ and $p_\theta = \partial_\theta S$. The number of mode with energy less than E and with a fixed m is obtained by integrating over p_θ in phase space

$$\begin{aligned}\Gamma(E,m) &= \frac{1}{\pi}\int d\phi d\theta \int_{r_+ + h}^{L} dr \int dp_\theta p_r(E,m,x) \qquad (25)\\ &= \frac{1}{\pi}\int d\phi d\theta \int_{r_+ + h}^{L} dr \int dp_\theta \left[\frac{1}{g^{rr}}\left(-g^{tt}E^2 + 2g^{t\phi}Em - g^{\phi\phi}m^2 - g^{\theta\theta}p_\theta^2 - \mu^2\right)\right]^{1/2}.\end{aligned}$$

The integration over p_θ must be carried out over the phase space that satisfies $p_r \geq 0$.

Note that if we identify m as p_ϕ then the rearrangement of the expression (24) yields Eq. (7). It is natural because WKB approximation means to treat the system as a classical one. In the WKB approximation the energy E satisfies the constraint (24). Thus in the region I, $E - \Omega_0 m > 0$. However in the region II it is possible that $E - \Omega_0 m < 0$. (It is a superradiance mode.) However as in classical case the geometry of the phase space for a given E is a non-compact surface. Thus the free energy diverges. In fact the surface such that ${g'}_{00} = 0$ is the velocity of light surface (VLS). Outside that surface a comoving observer must have a velocity $v_{ob} > 1$ and move on a spacelike world line.

In $\Omega_0 = \Omega_H$ case we can exactly find the position of the light of velocity surface. (In this case the region I corresponds to $r_+ < r < r_{VLS}$.) In such a case ${g'}_{tt}$ can be written as

$$\begin{aligned}{g'}_{tt} &= g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi} \qquad (26)\\ &= \frac{M^2}{\Sigma}(x - \bar{r}_+)\Big\{\bar{\Omega}_H^2 \sin^2\theta\; x^3 + \bar{r}_+\bar{\Omega}_H^2 \sin^2\theta\; x^2 \\ &\quad + \Big[-1 + \bar{\Omega}_H^2 \sin^2\theta\left(y^2 + y^2\cos^2\theta + \bar{r}_+^2\right)\Big]x + B\Big\} \\ &\equiv \frac{M^2}{\Sigma}(x - \bar{r}_+)\bar{\Omega}_H^2 \sin^2\theta\left(x^3 + a_1 x^2 + a_2 x + a_3\right) \qquad (27)\end{aligned}$$

for $\theta \neq 0$, where $x = r/M, y = a/M, z = e/M, \bar{\Omega}_H = M\Omega_H, \bar{r}_+ = r_+/M$, and

$$B = 2\left(1 - \bar{\Omega}_H y \sin^2\theta\right)^2 - \bar{r}_+ + \bar{r}_+\bar{\Omega}_H^2 \sin^2\theta\left(\bar{r}_+^2 + y^2 + y^2\cos^2\theta\right). \tag{28}$$

Then the exact position of the light of velocity surface is given by [14]

$$r_{light\ of\ velocity} \equiv r_{VLS} = 2M\sqrt{-Q}\cos\left(\frac{1}{3}\Theta\right) - \frac{1}{3}a_1 M, \tag{29}$$

where

$$\Theta = \arccos\left(\frac{R}{\sqrt{-Q^3}}\right) \tag{30}$$

with

$$Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}. \tag{31}$$

Eq. (29) approximately is given by

$$r_{VLS} \sim \frac{1}{\Omega_H \sin\theta} - \frac{r_+}{3}, \tag{32}$$

which is an open, roughly, cylindrical surface. For $\theta = 0$ it is always that $g'_{tt} < 0$ for $r > r_+$.

For extreme charged Kerr black hole case $M^2 = a^2 + e^2$ the position of VLS at $\theta = \frac{\pi}{2}$ is given by

$$r = M \qquad \text{for } \frac{1}{2}M \ \leq a \leq M \text{ and } a = 0, \tag{33}$$

$$r = \left(-1 + \frac{M}{a}\right)M \qquad \text{for } 0 < a < \frac{1}{2}M. \tag{34}$$

The second case corresponds to the extreme black hole that is slowly rotating and has many charge. (In this case $e > \sqrt{3}/2M \approx 0.866M$). In particular in case of $e \leq \sqrt{3}/2M$ ($a = M$ for $e = 0$) the horizon and the light of velocity surface are at the same position. Therefore in case of the extreme black hole with $a \geq 1/2M$ it is impossible to consider the brick wall model of 't Hooft.

Hereafter we assume that the outer brick wall is located inside the velocity of light surface, and that the black hole is not extrme. About the location for the outer brick wall (perfectly reflecting mirror) it was already pointed out in ref.[15] to remove the singular structure of the Hartle-Hawking vacuum state and to modify it.

The free energy is then written as

$$\begin{aligned}
\beta F &= \sum_m \int_{m\Omega_0}^{\infty} dE g(E,m) \ln\left(1 - e^{-\beta(E - m\Omega_0)}\right) \\
&= \int_0^{\infty} dE \sum_m g(E + m\Omega_0, m) \ln\left(1 - e^{-\beta E}\right) \\
&= -\beta \int_0^{\infty} dE \frac{1}{e^{\beta E} - 1} \int dm \Gamma(E + m\Omega_0, m),
\end{aligned} \tag{35}$$

where we have integrated by parts and we assume that the quantum number m is a continuous variable. The integrations over m and p_θ yield

$$F = -\frac{4}{3}\int d\phi d\theta \int_{r_+ + h}^{L} dr \int_{\mu\sqrt{-g'_{tt}}}^{\infty} dE \frac{1}{e^{\beta E}-1}\frac{\sqrt{g_4}}{\sqrt{-g'_{tt}}}\left(\frac{E^2}{-g'_{tt}}-\mu^2\right)^{3/2}, \tag{36}$$

where g'_{tt} can be written as

$$g'_{tt} = \frac{\mathcal{D}}{g_{\phi\phi}}\left[1-(\Omega-\Omega_0)^2\frac{g_{\phi\phi}^2}{-\mathcal{D}}\right]. \tag{37}$$

Note that the form of the free energy is similar to Eq. (11). In particular when $\Omega_0 = a = e = 0$, the free energy (36) coincides with the expression obtained by 't Hooft [7] and it is proportional to the volume of the optical space [10]. It is easy to see that the integrand diverges as $h \to 0$.

Let $\mu = 0$. For a massless scalar field the free energy reduces to

$$\beta F = -\frac{N}{\beta^3}\int d\theta d\phi \int_{r_+ + h}^{L} dr \frac{\sqrt{g_4}}{(-g'_{tt})^2} = -N\int_0^{\beta} d\tau \int d\theta d\phi \int_{r_+ + h}^{L} dr \sqrt{g_4}\frac{1}{\beta_{local}^4}, \tag{38}$$

where $\beta_{local} = \sqrt{-g'_{tt}}\beta$ is the reciprocal of the local Tolman temperature [19] and N is a constant. This form is just the free energy of a gas of massless particles at local temperature $1/\beta_{local}$. ¿From this expression (38) it is easy to obtain expressions for the total energy U, angular momentum J, and entropy S of a scalar field

$$J = \langle m\rangle = -\frac{1}{\beta}\frac{\partial}{\partial\Omega_0}(\beta F) = \frac{4N}{\beta^4}\int d\theta d\phi \int_{r_+ + h}^{L} dr \frac{\sqrt{g_4}}{(-g'_{tt})^2}\frac{g_{\phi\phi}}{(-g'_{tt})}(\Omega_0-\Omega), \tag{39}$$

$$U = \langle E\rangle = \Omega_0 J + \frac{\partial}{\partial\beta}(\beta F) = \frac{N}{\beta^4}\int d\theta d\phi \int_{r_+ + h}^{L} dr \frac{\sqrt{g_4}}{(-g'_{tt})^2}\left[3+4\frac{\Omega_0(\Omega_0-\Omega)g_{\phi\phi}}{(-g'_{tt})}\right], \tag{40}$$

$$S = \beta^2\frac{\partial}{\partial\beta}F = \beta(U-F-\Omega_0 J) = 4\frac{N}{\beta^3}\int d\theta d\phi \int_{r_+ + h}^{L} dr \frac{\sqrt{g_4}}{(-g'_{tt})^2}. \tag{41}$$

Now let us assume that $\Omega_0 \sim \Omega_H$, i.e. the scalar field is co-rotating with the black hole. Then near the event horizon $r = r_+$, the leading behavior of the free energy F for small h is

$$\begin{aligned}\beta F &\approx -\frac{N}{\beta^3}\int d\phi d\theta \int_{r_+ + h}^{L} dr \sqrt{g_4}\left(\frac{g_{\phi\phi}}{-\mathcal{D}}\right)^2 \\ &= -\frac{N}{\beta^3}\int d\phi d\theta \int_{r_+ + h}^{L} dr \sqrt{g_{\theta\theta}g_{\phi\phi}}\sqrt{g_{rr}}\left(\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}\right)^{3/2} \\ &\approx -N\frac{1}{2(\kappa\beta)^3}\frac{A_H}{\epsilon^2},\end{aligned} \tag{42}$$

where A_H is the area of the event horizon. The leading behaviors of the total angular momentum J, energy U and entropy S are

$$
\begin{aligned}
J &= 0, && (43)\\
U &= 3N\frac{1}{2\kappa^3\beta^4}\frac{A_H}{\epsilon^2}, && (44)\\
S &= 4N\frac{1}{2(\kappa\beta)^3}\frac{A_H}{\epsilon^2}. && (45)
\end{aligned}
$$

The leading behaviors of the thermodynamical quantities U, and S are divergent as $h \to 0$. But the angular momentum J is 0. The divergences arise because the phase volume $\Gamma(E) = \int dm\Gamma(E + m\Omega_0, m)$ diverges as h goes to zero. Actually the phase volume $\Gamma(E)$ is the same to the classical one $\Gamma_{cl}(E)$.

If we take T as the Hartle-Hawking temperature $T_H = \frac{\kappa}{2\pi}$ (In this case the quantum state is the Hartle-Hawking vacuum state $|H\rangle$ [18].) the entropy becomes

$$S = N'\frac{A_H}{\epsilon^2}, \tag{46}$$

where N' is a new constant. The entropy of a scalar field diverges quadratically in ϵ^{-1} as the system approaches to the horizon. Our result (46) agrees with the result calculated by 't Hooft [7]. This fact implies that the leading behavior of entropy (46) is general form.

4 Discussion

We have calculated the entropies of the systems of classical particles and a quantum field at thermal equilibrium with temperature T in the charged Kerr black hole. The leading behavior of the entropy of a quantum field is proportional to the area of the event horizon. But the classical entropy does not proportional to it. Such leading forms of the entropies can be also easily calculated by studying the asymptotic behavior of the metric (4) near the horizon.

Consider a locally co-rotating observer at a point near the horizon, who carries the following orthonormal frame

$$ds^2 = -(\kappa\rho)^2 dt^2 + g_{\phi\phi}(r = r_+, \theta)d\phi'^2 + d\rho^2 + g_{\theta\theta}(r = r_+, \theta)d\theta^2. \tag{47}$$

Then the locally measured energy by him is given by $E_{loc} = E/(\kappa\rho)$. Therefore he will think that the allowed momentum space volume is given by

$$V_{loc} = \frac{4\pi}{3}\left[E_{loc}^2 - \mu^2\right]^{3/2} = \frac{4\pi}{3}\left[\frac{E^2}{(\kappa\rho)^2} - \mu^2\right]^{3/2}, \tag{48}$$

and the total phase volume is

$$\begin{aligned} \Gamma(E) &= \int d\theta d\phi d\rho \sqrt{g_{\phi\phi} g_{\theta\theta}} \;\; V_{loc} \\ &= \frac{4\pi}{3}\int d\theta d\phi \sqrt{g_{\phi\phi} g_{\theta\theta}} \int d\rho \left[\frac{E}{\kappa\rho}\right]^3 \\ &= \frac{2\pi}{3} E^3 \frac{A_H}{\kappa^3 \epsilon^2} \end{aligned} \tag{49}$$

for $\mu = 0$. ¿From this one can obtain the free energy (42) and the entropies (17), (45). Thus the fundamental reason of divergences is the infinite number of state or the infinite volume of the phase space near the horizon. As in classical case, it can be attributed to the infinite blue shift of the energy E.

In case of the spherically symmetric black hole, we need the outer brick wall at $r = L$ to eliminate the infra-red divergence. However in the case of the rotating black hole we need the outer brick wall at $r = L$, where $L < r_{VLS}$, to prevent a system from having the velocity greater than the velocity of light. If not, all thermodynamical quantities are divergent. (It is related to the singularity of the Hartle-Hawking state $|H\rangle$ [15].) If we consider a rotating system in flat space, such a fact becomes more apparent. In WKB approximation the free energy of the rotating system (see (36)) is given by

$$\beta F = -\frac{N}{\beta^3}\int d\phi dz \int_{region\ I} \frac{r}{(1 - \Omega_0^2 r^2)^2} dr \tag{50}$$

in the cylindrical coordinates. As $r \to 1/\Omega_0$ (the velocity of light surface) the free energy diverges. Outside the surface of the velocity of light (in the region II) the free energy is divergent because of the infinite volume of phase space. Thus we must restrict a system in the region I.

The particular point is that in case of the extreme rotating black such that $a \geq 1/2M$ and $\Omega_0 = \Omega_H$ we can not consider the brick wall model of 't Hooft. This point is the different point with other extreme black hole like Reissner-Nordstrom one.

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